ROOT-SUM-OF-SQUARES CIRCUITS WITH THE 433

by Dan Sheingold

In Volume 6, No. 2, the Model 433* multifunction analog module was introduced. It is a device that is characterized by the function: $(10/V_R)Y(Z/X)^m$, where all inputs and the output can have any value from 0 to +10V, $V_R \cong 9V$, and the exponent m can range from 0.2 to 5, as set by a resistance ratio. The useful jobs it can do at low cost include multiplication, division (ratios), exponentiation, rms computation, and vector sums.

In this Brief, use of the 433 for vector sums will be discussed at greater length. In a second Brief, in the following pages, we will touch on some new ideas developed by the author involving a few interesting circuits to obtain economical (but reasonably accurate) embodiments of trigonometric relationships.

$\sqrt{X^2 + Y^2}$: TWO WAYS

The 433 is used with two operational amplifiers to fulfill the relationship

$$(V_C + V_B)(V_C - V_B) = V_A^2 \tag{1}$$

whence

$$V_C = \sqrt{V_A^2 + V_B^2} \tag{2}$$

In concept, the equation may be embodied in two ways:

$$V_C = \frac{V_A^2}{V_C + V_B} + V_B \tag{3}$$

and

$$V_C = \frac{V_A^2}{V_C - V_B} - V_B \tag{4}$$

In practice, the circuit embodying equation (3) leads to better results (Figure 1). The denominator is always at least as large as input V_A; hence the closed-loop gain is $\leqslant 1$. The result is excellent stability and low noise, even for very small inputs.

Unfortunately, a circuit embodying equation (4) has appeared in some of our earlier literature through the appalling workings of Murphy's Law ("the worse of two approaches will always be published"). It can be made to work, but at the cost of large stabilizing capacitors, reduced bandwidth, and poor accuracy for low-level inputs.

Performance, adjustment, and choice of components are straight-forward. A feature of the circuit that does require some thought is the choice of input scaling, to avoid saturation of devices within the circuit. Figure 2 shows the "worst-case" situation, with specified 10V limits on all amplifier outputs. The "gating" factor is the output of amplifier A1, which takes the sum $V_C + V_B$. For the output of A1 to remain within the 10V limit, the maximum values of V_A and V_B must lie within the envelope shown in the Figure. For the particular case of $V_A = V_B$, the inputs should not exceed 4.142V (output = 5.858V). As a practical matter, IC amplifiers are usually specified to have 12V minimum output with light loads: for both inputs at 5V (a "round" number) the output is 7.07V, and the output of A1 is 12.07V. With careful matching and adjustment, one can typically achieve errors less than 0.25% of the theoretical output for inputs from 0.1 to 10V.

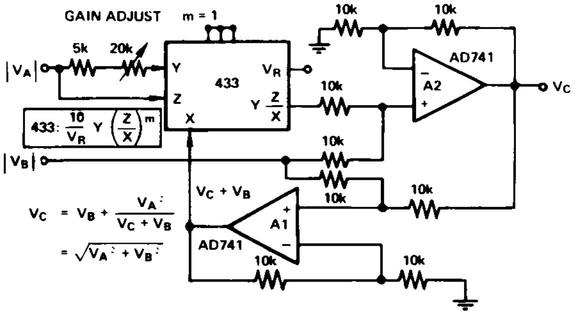


Figure 1. Preferred Circuit to Compute $\sqrt{V_A^2 + V_B^2}$

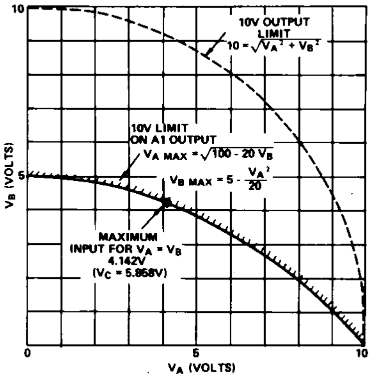


Figure 2. Input Voltage Limits (Using Amplifiers With 10V Output Specifications)

MORE THAN TWO INPUTS

This unconventional approach to square-root of the sum-of-squares can be used for the more-general case of 3 or more input variables without the necessity of cascading operations on pairs of inputs. Simply sum n-1 numerator terms, i.e.,

$$E_0 = \frac{V_1^2 + V_2^2 + \ldots V_{n-1}^2}{E_0 + V_n} + V_n \tag{5}$$

$$= \sqrt{V_1^2 + V_2^2 + \ldots V_n^2} \tag{6}$$

Figure 3 is a scheme that embodies this approach. Again, the principal constraint is imposed by the output specifications of amplifier A1. If all inputs have the same maximum value (and can have it simultaneously), for the maximum output of A1 to be less than E_{max}, that maximum input value V_{max} is

$$V_{max} = E_{max} / (1 + \sqrt{n}) \tag{7}$$

For E_{max} = 10V, corresponding values of n and V_{max} are *2* : 4.14V, *3* : 3.66V, *4* : 3.33V, *5* : 3.09V, *6* : 2.90V, etc. ▶▶▶

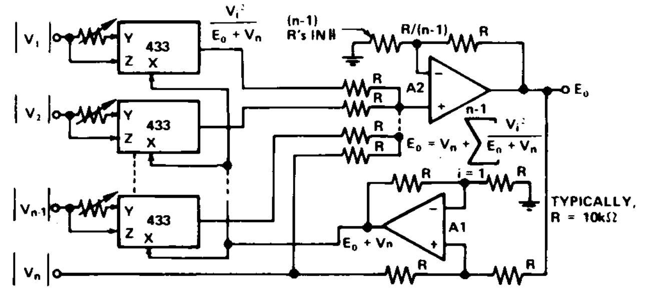


Figure 3. Extension of the Technique to n Input Signals

Application Briefs

TRIGONOMETRIC OPERATIONS WITH THE 433

(The Powers of Non-Integral Exponents)

by Dan Sheingold

A new approximation technique appears to presage promising results in the low-cost analog simulation of trigonometric relationships. Applications include low-cost resolvers, coordinate transformations (especially Cartesian-to-polar), and generation of time functions. Further extensions of the basic idea will be useful in such applications as "linearizing" the outputs of transducers having nonlinear distortion (thermocouples and strain gages, for example) and the generation of arbitrary functions having no discontinuities (as compared to functions simulated by piecewise-linear approximations).

The results reported on are preliminary and subject to refinement by the author, as well as by readers having more considerable mathematical talents. While the application of the technique is entirely new in our experience, there may have been precursors in the literature, of which we are unaware. (We would appreciate enlightenment in this regard.)

The basic idea, the use of non-integral exponents, has been germinating in the back of our mind since the days of the obsolete (35+ watts) GAP/R K4-FG polynomial Function Generator (1951). Two important conditions were required to bring the idea to the fore:

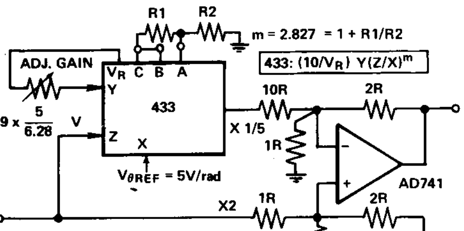
1. The availability of a low-cost, reasonably-accurate multiplier-divider component incorporating arbitrary and easily-adjustable exponents, to wit, the present Analog Devices Model 433 (see preceding page).
2. The availability of computing devices capable of performing large numbers of transcendental computations in a short period of time, for evaluation of approximation schemes. (Bouquets to the HP-35!)

SIN θ

An example that illustrates the power of the technique is the embodiment of sin θ, to within ¼% in a single quadrant. Most of our readers are familiar with the infinite series for sin θ

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \qquad (1)$$

Approximations using modified coefficients allow truncation of the series to 3 terms (θ^5) with less than 0.02% error[1], and to 2 terms with 1.35% error.



$$E_o = 10\left(\frac{V_\theta}{V_{REF}}\right) - \frac{10}{6.28}\left(\frac{V_\theta}{V_{REF}}\right)^{2.827} \cong 10\sin\left(\frac{V_\theta}{V_{REF}}\right)$$

Figure 1. Basic Circuit for Approximating sin θ

However, by using non-integral exponents, in addition to modified coefficients, one can obtain less than 0.25% error (one quadrant) with just two terms! A reasonably close approximation is

$$\sin\theta \cong \theta - \frac{\theta^{2.827}}{6.28} \qquad (\,0 \text{ to } \pi/2 \text{ radians}) \qquad (2)$$

It can be implemented with a simple circuit composed of a 433 and a single operational amplifier. A circuit and a plot of the theoretical errors will be found in Figures 1 and 2.

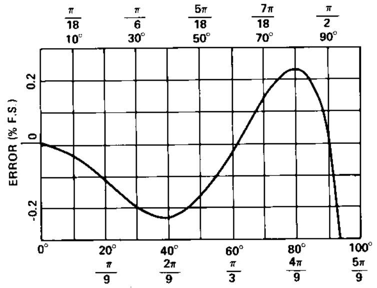
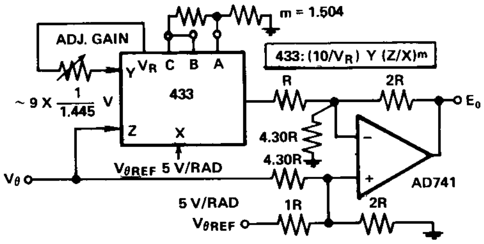


Figure 2. Theoretical Errors of sin θ Approximation (% F.S.)

For "round-number" scaling, one might typically choose either 5V/radian or 1V/10°. Voltage equivalents for both are given in the table.

Radians	Degrees	Volts	Degrees	Radians	Volts
0	0	0	0	0	0
0.1	5.73	0.5	10	0.1745	1
0.2	11.46	1.0	20	0.3491	2
0.3	17.19	1.5	30	0.5236	3
0.4	22.92	2.0	40	0.6981	4
0.5	28.65	2.5	45	0.7854	4.5
0.6	34.38	3.0	50	0.8727	5
0.7	40.11	3.5	60	1.047	6
0.8	45.84	4.0	70	1.222	7
0.9	51.57	4.5	80	1.396	8
1.0	57.296	5.0	90	1.5708	9
1.1	63.03	5.5	100	1.745	10
1.2	68.75	6.0			
1.3	74.48	6.5			
1.4	80.21	7.0			
1.5	85.94	7.5			
1.6	91.67	8.0			
1.7	97.40	8.5			



$$\frac{E_o}{10} = 1 + 0.2325\left(\frac{V_\theta}{V_{REF}}\right) - \frac{1}{1.445}\left(\frac{V_\theta}{V_{REF}}\right)^{1.504} \cong \cos\frac{V_\theta}{V_{REF}}$$

Figure 3. Basic Circuit for Approximating cos θ

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[1] See *Handbook of Mathematical Functions,* U.S. Department of Commerce, National Bureau of Standards, edited by Abramowitz and Stegun, 1968, available from the Superintendent of Documents, Washington, D.C. 20402 (1046 pp.).

Application Briefs

COS θ

For approximating cos θ to reasonable accuracy, two terms are inadequate. However, by using arbitrary exponents and a *linear* third term, we can get a better-than-1% approximation using only a single power term. Again, the function can be embodied with a single 433 and a single op amp. A reasonably close approximation* is

$$\cos\theta \cong 1 + 0.2325\,\theta - \frac{\theta^{1.504}}{1.445} \qquad (3)$$

A circuit and a plot of the theoretical errors can be found in Figures 3 and 4.

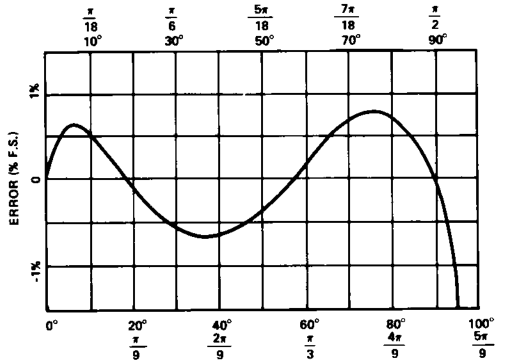

Figure 4. Theoretical Errors of cos θ Approximation (% F.S.)

TAN⁻¹ (V_B/V_A)

The arctangent is inherently one of the most difficult functions to fit because of the wide dynamic range that the input ratio ideally must cover. "Building-block" approximations generally have limited feasibility because the ratio must appear explicitly as the result of a division.

Yet the function is useful: it provides the angular information in the transformation $R = \sqrt{V_A^2 + V_B^2}$, $\theta = \tan^{-1}(V_B/V_A)$. We have already seen (p. 3) how to obtain R efficiently.

It yields under the combined assault of

1. The 433, used to obtain a non-integral exponent
2. Feedback techniques to obtain an equation which is an implicit solution for V_θ (to a very good approximation)
3. The 433's ability to convert a widely-ranging ratio into a harmless difference of logarithms in its internal workings, without explicitly obtaining the ratio
4. Ever-present Mother Nature, who is always ready to help us, if only we will listen!

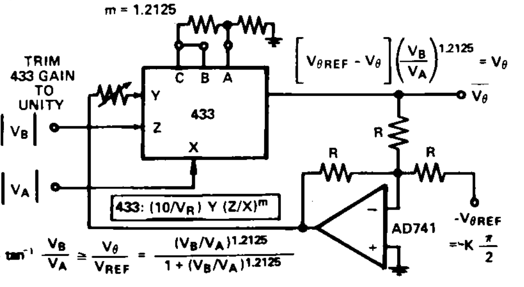

Figure 5. Basic Circuit for Approximating the Arctangent of a Ratio.

A circuit that embodies the approximation is shown in Figure 5. The theoretical errors are plotted in Figure 6. The equation fundamentally provided by the scheme is

$$\theta = 90^\circ \frac{W^{1.2125}}{1 + W^{1.2125}} \cong \tan^{-1} W \qquad (4)$$

where $W = V_B/V_A$

The circuit solves the implicit equation

$$\theta = \left[\frac{\pi}{2} - \theta\right]\left[\frac{V_B}{V_A}\right]^{1.2125} \qquad (5)$$

with a maximum theoretical error less than 0.75% $\frac{\pi}{2}$ (or 0.68°).

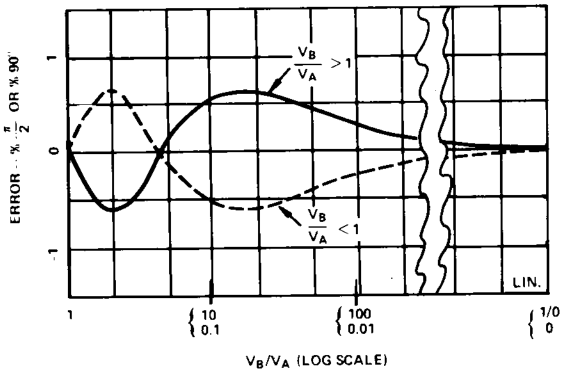

Figure 6. Theoretical Errors of Arctan Approximation

PRACTICAL CONSIDERATIONS

The numbers presented above have all been worked out to about 4 places to determine the limits of performance with ideal circuitry. However, since performance depends critically on the analog circuit elements, care must be observed to take into account component tolerances, and provide "tweaks" for an empirical fit. In the case of the arctangent approximation, care may be necessary to avoid oscillation at large values of the ratio. Large values of feedback capacitance around the amplifier will be helpful, but will slow the circuit's response.

TABULAR DATA

The table presents ideal and approximate values of the functions discussed above, for the typical coefficient values listed. Although there may be coefficient values that would give somewhat better fit, they are perhaps best determined empirically, in view of the tolerances of the 433 and the resistor ratios. The table will help in such empirical determinations.

θ°	sin θ		cos θ		W	tan⁻¹ W		
	Ideal	Approx.	Ideal	Approx.		Ideal°	Ideal (r)	Approx.
0	0.	0.	1.	1.	100	89.43	1.5608	1.5649
10	0.1736	0.1734	0.9848	0.9905	30	88.09	1.5375	1.5458
20	0.3420	0.3409	0.9397	0.9390	10	84.29	1.4711	1.4801
30	0.5	0.4980	0.8660	0.8602	3	71.57	1.2490	1.2428
40	0.6428	0.6405	0.7660	0.7591	1	45.	0.7854	0.7854
50	0.7660	0.7643	0.6428	0.6389	0.5	26.57	0.4636	0.4735
60	0.8660	0.8658	0.5	0.5016	0.2	11.31	0,1974	0.1954
70	0.9397	0.9412	0.3420	0.3485	0.1	5.71	0.0997	0.0907
80	0.9848	0.9871	0.1736	0.1810	0.05	2.86	0.05	0.0405
90	1.	1.	0.	0.	0.	0.	0.	0.

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*A more-accurate one (using 2 op amps) is

$$\cos\theta = \sin\left[\frac{\pi}{2} - \theta\right] \cong \left[\frac{\pi}{2} - \theta\right] - \frac{1}{6.28}\left[\frac{\pi}{2} - \theta\right]^{2.827} \quad (3a)$$