

The Complex Fourier Transform

Although complex numbers are fundamentally disconnected from our reality, they can be used to solve science and engineering problems in two ways. First, the parameters from a real world problem can be substituted into a complex form, as presented in the last chapter. The second method is much more elegant and powerful, a way of making the complex numbers mathematically equivalent to the physical problem. This approach leads to the *complex Fourier transform*, a more sophisticated version of the *real Fourier transform* discussed in Chapter 8. The complex Fourier transform is important in itself, but also as a stepping stone to more powerful complex techniques, such as the *Laplace* and *z-transforms*. These complex transforms are the foundation of theoretical DSP.

The Real DFT

All four members of the Fourier transform family (DFT, DTFT, Fourier Transform & Fourier Series) can be carried out with either real numbers or complex numbers. Since DSP is mainly concerned with the DFT, we will use it as an example. Before jumping into the complex math, let's review the real DFT with a special emphasis on things that are awkward with the mathematics. In Chapter 8 we defined the *real* version of the Discrete Fourier Transform according to the equations:

EQUATION 31-1

The real DFT. This is the forward transform, calculating the frequency domain from the time domain. In spite of using the names: *real part* and *imaginary part*, these equations only involve ordinary numbers. The frequency index, k , runs from 0 to $N/2$. These are the same equations given in Eq. 8-4, except that the $2/N$ term has been included in the forward transform.

$$Re X[k] = \frac{2}{N} \sum_{n=0}^{N-1} x[n] \cos(2\pi kn/N)$$

$$Im X[k] = \frac{-2}{N} \sum_{n=0}^{N-1} x[n] \sin(2\pi kn/N)$$

In words, an N sample time domain signal, $x[n]$, is decomposed into a set of $N/2 + 1$ cosine waves, and $N/2 + 1$ sine waves, with frequencies given by the

index, k . The amplitudes of the cosine waves are contained in $ReX[k]$, while the amplitudes of the sine waves are contained in $ImX[k]$. These equations operate by *correlating* the respective cosine or sine wave with the time domain signal. In spite of using the names: *real part* and *imaginary part*, there are no complex numbers in these equations. There isn't a j anywhere in sight! We have also included the normalization factor, $2/N$ in these equations. Remember, this can be placed in front of either the synthesis or analysis equation, or be handled as a separate step (as described by Eq. 8-3). These equations should be very familiar from previous chapters. If they aren't, go back and brush up on these concepts before continuing. If you don't understand the *real* DFT, you will never be able to understand the *complex* DFT.

Even though the real DFT uses only real numbers, *substitution* allows the frequency domain to be *represented* using complex numbers. As suggested by the names of the arrays, $ReX[k]$ becomes the real part of the complex frequency spectrum, and $ImX[k]$ becomes the imaginary part. In other words, we place a j with each value in the imaginary part, and add the result to the real part. However, do not make the mistake of thinking that this is the "complex DFT." This is nothing more than the *real DFT with complex substitution*.

While the real DFT is adequate for many applications in science and engineering, it is mathematically awkward in three respects. First, it can only take advantage of complex numbers through the use of **substitution**. This makes mathematicians uncomfortable; they want to say: "this *equals* that," not simply: "this *represents* that." For instance, imagine we are given the mathematical statement: A equals B . We immediately know countless consequences: $5A = 5B$, $1+A = 1+B$, $A/x = B/x$, etc. Now suppose we are given the statement: A represents B . Without additional information, we know absolutely nothing! When things are equal, we have access to four-thousand years of mathematics. When things only represent each other, we must start from scratch with new definitions. For example, when sinusoids are represented by complex numbers, we allow addition and subtraction, but prohibit multiplication and division.

The second thing handled poorly by the real Fourier transform is the **negative frequency** portion of the spectrum. As you recall from Chapter 10, sine and cosine waves can be described as having a *positive* frequency or a *negative* frequency. Since the two views are identical, the real Fourier transform ignores the negative frequencies. However, there are applications where the negative frequencies are important. This occurs when negative frequency components are forced to move into the positive frequency portion of the spectrum. The ghosts take human form, so to speak. For instance, this is what happens in aliasing, circular convolution, and amplitude modulation. Since the real Fourier transform doesn't use negative frequencies, its ability to deal with these situations is very limited.

Our third complaint is the **special handling of $ReX[0]$ and $ReX[N/2]$** , the first and last points in the frequency spectrum. Suppose we start with an N

point signal, $x[n]$. Taking the DFT provides the frequency spectrum contained in $ReX[k]$ and $ImX[k]$, where k runs from 0 to $N/2$. However, these are not the amplitudes needed to reconstruct the time domain waveform; samples $ReX[0]$ and $ReX[N/2]$ must first be divided by two. (See Eq. 8-3 to refresh your memory). This is easily carried out in computer programs, but inconvenient to deal with in equations.

The complex Fourier transform is an elegant solution to these problems. It is natural for complex numbers and negative frequencies to go hand-in-hand. Let's see how it works.

Mathematical Equivalence

Our first step is to show how sine and cosine waves can be written in an *equation* with complex numbers. The key to this is Euler's relation, presented in the last chapter:

EQUATION 31-2
Euler's relation.

$$e^{jx} = \cos(x) + j \sin(x)$$

At first glance, this doesn't appear to be much help; one complex expression is equal to another complex expression. Nevertheless, a little algebra can rearrange the relation into two other forms:

EQUATION 31-3
Euler's relation for
sine & cosine.

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \quad \sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

This result is extremely important, we have developed a way of writing *equations* between complex numbers and ordinary sinusoids. Although Eq. 31-3 is the standard form of the identity, it will be more useful for this discussion if we change a few terms around:

EQUATION 31-4
Sinusoids as complex numbers. Using complex numbers, cosine and sine waves can be written as the sum of a positive and a negative frequency.

$$\begin{aligned} \cos(\omega t) &= \frac{1}{2} e^{j(-\omega)t} + \frac{1}{2} e^{j\omega t} \\ \sin(\omega t) &= \frac{1}{2} j e^{j(-\omega)t} - \frac{1}{2} j e^{j\omega t} \end{aligned}$$

Each expression is the sum of two exponentials: one containing a *positive* frequency (ω), and the other containing a *negative* frequency ($-\omega$). In other words, when sine and cosine waves are written as complex numbers, the

negative portion of the frequency spectrum is automatically included. The positive and negative frequencies are treated with an equal status; it requires one-half of each to form a complete waveform.

The Complex DFT

The forward complex DFT, written in polar form, is given by:

EQUATION 31-5

The forward complex DFT. Both the time domain, $x[n]$, and the frequency domain, $X[k]$, are arrays of complex numbers, with k and n running from 0 to $N-1$. This equation is in polar form, the most common for DSP.

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

Alternatively, Euler's relation can be used to rewrite the forward transform in rectangular form:

EQUATION 31-6

The forward complex DFT (rectangular form).

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \left(\cos(2\pi kn/N) - j \sin(2\pi kn/N) \right)$$

To start, compare this equation of the *complex Fourier transform* with the equation of the *real Fourier transform*, Eq. 31-1. At first glance, they appear to be identical, with only small amount of algebra being required to turn Eq. 31-6 into Eq. 31-1. However, this is very misleading; the differences between these two equations are very subtle and easy to overlook, but tremendously important. Let's go through the differences in detail.

First, the real Fourier transform converts a real time domain signal, $x[n]$, into two real frequency domain signals, $ReX[k]$ & $ImX[k]$. By using complex substitution, the frequency domain can be *represented* by a single complex array, $X[k]$. In the complex Fourier transform, both $x[n]$ & $X[k]$ are arrays of complex numbers. A practical note: Even though the time domain is complex, there is nothing that *requires* us to use the imaginary part. Suppose we want to process a real signal, such as a series of voltage measurements taken over time. This group of data becomes the real part of the time domain signal, while the imaginary part is composed of zeros.

Second, the real Fourier transform only deals with *positive* frequencies. That is, the frequency domain index, k , only runs from 0 to $N/2$. In comparison, the complex Fourier transform includes both *positive* and *negative* frequencies. This means k runs from 0 to $N-1$. The frequencies between 0 and $N/2$ are positive, while the frequencies between $N/2$ and $N-1$ are negative. Remember, the frequency spectrum of a discrete signal is periodic, making the negative frequencies between $N/2$ and $N-1$ the same as

between $-N/2$ and 0. The samples at 0 and $N/2$ straddle the line between positive and negative. If you need to refresh your memory on this, look back at Chapters 10 and 12.

Third, in the real Fourier transform with substitution, a j was added to the sine wave terms, allowing the frequency spectrum to be represented by complex numbers. To convert back to ordinary sine and cosine waves, we can simply drop the j . This is the sloppiness that comes when one thing only *represents* another thing. In comparison, the complex DFT, Eq. 31-5, is a formal mathematical equation with j being an integral part. In this view, we cannot arbitrarily add or remove a j any more than we can add or remove any other variable in the equation.

Forth, the real Fourier transform has a scaling factor of *two* in front, while the complex Fourier transform does not. Say we take the real DFT of a cosine wave with an amplitude of *one*. The spectral value corresponding to the cosine wave is also *one*. Now, let's repeat the process using the complex DFT. In this case, the cosine wave corresponds to *two* spectral values, a positive and a negative frequency. Both these frequencies have a value of $1/2$. In other words, a positive frequency with an amplitude of $1/2$, combines with a negative frequency with an amplitude of $1/2$, producing a cosine wave with an amplitude of *one*.

Fifth, the real Fourier transform requires special handling of two frequency domain samples: $ReX[0]$ & $ReX[N/2]$, but the complex Fourier transform does not. Suppose we start with a time domain signal, and take the DFT to find the frequency domain signal. To reverse the process, we take the Inverse DFT of the frequency domain signal, reconstructing the original time domain signal. However, there is scaling required to make the reconstructed signal be identical to the original signal. For the complex Fourier transform, a factor of $1/N$ must be introduced somewhere along the way. This can be tacked-on to the forward transform, the inverse transform, or kept as a separate step between the two. For the real Fourier transform, an additional factor of two is required ($2/N$), as described above. However, the real Fourier transform also requires an additional scaling step: $ReX[0]$ and $ReX[N/2]$ must be divided by *two* somewhere along the way. Put in other words, a scaling factor of $1/N$ is used with these two samples, while $2/N$ is used for the remainder of the spectrum. As previously stated, this awkward step is one of our complaints about the real Fourier transform.

Why are the real and complex DFTs different in how these two points are handled? To answer this, remember that a cosine (or sine) wave in the time domain becomes split between a positive and a negative frequency in the complex DFT's spectrum. However, there are two exceptions to this, the spectral values at 0 and $N/2$. These correspond to zero frequency (DC) and the Nyquist frequency (one-half the sampling rate). Since these points straddle the positive and negative portions of the spectrum, they do not have a matching point. Because they are not combined with another value, they inherently have only one-half the contribution to the time domain as the other frequencies.

FIGURE 31-1

Complex frequency spectrum. These curves correspond to an entirely real time domain signal, because the real part of the spectrum has an *even* symmetry, and the imaginary part has an *odd* symmetry. The two square markers in the real part correspond to a cosine wave with an amplitude of one, and a frequency of 0.23. The two round markers in the imaginary part correspond to a sine wave with an amplitude of one, and a frequency of 0.23.

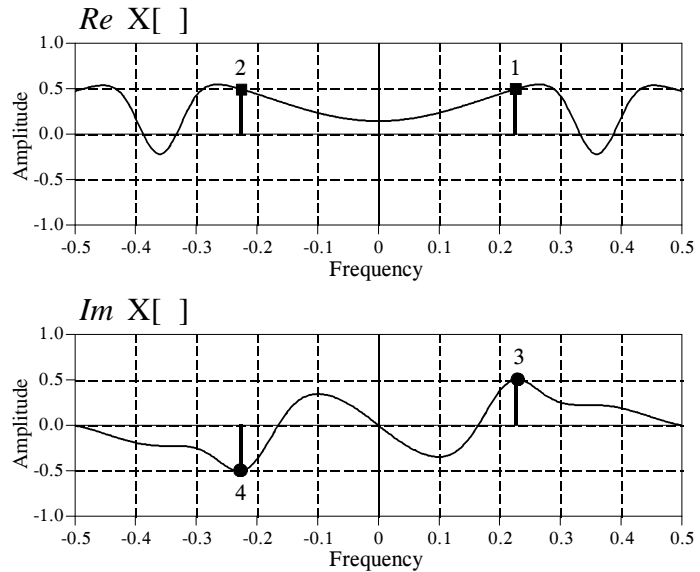


Figure 31-1 illustrates the complex DFT's frequency spectrum. This figure assumes the time domain is entirely *real*, that is, its imaginary part is zero. We will discuss the idea of imaginary time domain signals shortly. There are two common ways of displaying a complex frequency spectrum. As shown here, zero frequency can be placed in the center, with positive frequencies to the right and negative frequencies to the left. This is the best way to *think* about the complete spectrum, and is the *only* way that an aperiodic spectrum can be displayed.

The problem is that the spectrum of a discrete signal is *periodic* (such as with the DFT and the DTFT). This means that everything between -0.5 and 0.5 repeats itself an infinite number of times to the left and to the right. In this case, the spectrum between 0 and 1.0 contains the same information as from -0.5 to 0.5. When graphs are made, such as Fig. 31-1, the -0.5 to 0.5 convention is usually used. However, many equations and programs use the 0 to 1.0 form. For instance, in Eqs. 31-5 and 31-6 the frequency index, k , runs from 0 to $N-1$ (coinciding with 0 to 1.0). However, we could write it to run from $-N/2$ to $N/2-1$ (coinciding with -0.5 to 0.5), if we desired.

Using the spectrum in Fig. 31-1 as a guide, we can examine how the inverse complex DFT reconstructs the time domain signal. The inverse complex DFT, written in polar form, is given by:

EQUATION 31-7

The inverse complex DFT. This is matching equation to the forward complex DFT in Eq. 31-5.

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$$

Using Euler's relation, this can be written in rectangular form as:

EQUATION 31-8
The inverse complex DFT.
This is Eq. 31-7 rewritten to
show how each value in the
frequency spectrum affects
the time domain.

$$x[n] = \sum_{k=0}^{N-1} \text{Re}X[k] \left(\cos(2\pi kn/N) + j \sin(2\pi kn/N) \right) - \sum_{k=0}^{N-1} \text{Im}X[k] \left(\sin(2\pi kn/N) - j \cos(2\pi kn/N) \right)$$

The compact form of Eq. 31-7 is how the inverse DFT is usually written, although the expanded version in Eq. 31-9 can be easier to understand. In words, each value in the real part of the frequency domain contributes a real cosine wave *and* an imaginary sine wave to the time domain. Likewise, each value in the imaginary part of the frequency domain contributes a real sine wave *and* an imaginary cosine wave. The time domain is found by adding all these real and imaginary sinusoids. The important concept is that each value in the frequency domain produces both a *real* sinusoid and an *imaginary* sinusoid in the time domain.

For example, imagine we want to reconstruct a unity amplitude cosine wave at a frequency of $2\pi k/N$. This requires a positive frequency and a negative frequency, both from the real part of the frequency spectrum. The two square markers in Fig. 31-1 are an example of this, with the frequency set at: $k/N = 0.23$. The positive frequency at 0.23 (labeled 1 in Fig. 31-1) contributes a cosine wave and an imaginary sine wave to the time domain:

$$\frac{1}{2} \cos(2\pi 0.23 n) + \frac{1}{2} j \sin(2\pi 0.23 n)$$

Likewise, the negative frequency at -0.23 (labeled 2 in Fig. 31-1) also contributes a cosine and an imaginary sine wave to the time domain:

$$\frac{1}{2} \cos(2\pi (-0.23) n) + \frac{1}{2} j \sin(2\pi (-0.23) n)$$

The negative sign within the cosine and sine terms can be eliminated by the relations: $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$. This allows the negative frequency's contribution to be rewritten:

$$\frac{1}{2} \cos(2\pi 0.23 n) - \frac{1}{2} j \sin(2\pi 0.23 n)$$

Adding the contributions from the positive and the negative frequencies reconstructs the time domain signal:

$$\begin{array}{l}
 \text{contribution from positive frequency} \rightarrow \quad \frac{1}{2} \cos(2\pi 0.23 n) + \frac{1}{2} j \sin(2\pi 0.23 n) \\
 \text{contribution from negative frequency} \rightarrow \quad \frac{1}{2} \cos(2\pi 0.23 n) - \frac{1}{2} j \sin(2\pi 0.23 n) \\
 \hline
 \text{resultant time domain signal} \rightarrow \quad \cos(2\pi 0.23 n)
 \end{array}$$

In this same way, we can synthesize a sine wave in the time domain. In this case, we need a positive and negative frequency from the imaginary part of the frequency spectrum. This is shown by the round markers in Fig. 31-1. From Eq. 31-8, these spectral values contribute a sine wave and an imaginary cosine wave to the time domain. The imaginary cosine waves cancel, while the real sine waves add:

$$\begin{array}{l}
 \text{contribution from positive frequency} \rightarrow \quad -\frac{1}{2} \sin(2\pi 0.23 n) - \frac{1}{2} j \cos(2\pi 0.23 n) \\
 \text{contribution from negative frequency} \rightarrow \quad -\frac{1}{2} \sin(2\pi 0.23 n) + \frac{1}{2} j \cos(2\pi 0.23 n) \\
 \hline
 \text{resultant time domain signal} \rightarrow \quad -\sin(2\pi 0.23 n)
 \end{array}$$

Notice that a *negative* sine wave is generated, even though the positive frequency had a value that was *positive*. This sign inversion is an inherent part of the mathematics of the complex DFT. As you recall, this same sign inversion is commonly used in the real DFT. That is, a *positive* value in the imaginary part of the frequency spectrum corresponds to a *negative* sine wave. Most authors include this sign inversion in the definition of the real Fourier transform to make it consistent with its complex counterpart. The point is, this sign inversion *must* be used in the complex Fourier transform, but is merely an option in the real Fourier transform.

The *symmetry* of the complex Fourier transform is very important. As illustrated in Fig. 31-1, a real time domain signal corresponds to a frequency spectrum with an *even* real part, and an *odd* imaginary part. In other words, the negative and positive frequencies have the same sign in the real part (such as points 1 and 2 in Fig. 31-1), but opposite signs in the imaginary part (points 3 and 4).

This brings up another topic: the imaginary part of the time domain. Until now we have assumed that the time domain is completely real, that is, the imaginary part is zero. However, the complex Fourier transform does not require this.

What is the physical meaning of an imaginary time domain signal? Usually, there is none. This is just something allowed by the complex mathematics, without a correspondence to the world we live in. However, there are applications where it can be used or manipulated for a mathematical purpose.

An example of this is presented in Chapter 12. The imaginary part of the time domain produces a frequency spectrum with an odd real part, and an even imaginary part. This is just the opposite of the spectrum produced by the real part of the time domain (Fig. 31-1). When the time domain contains both a real part and an imaginary part, the frequency spectrum is the sum of the two spectra, had they been calculated individually. Chapter 12 describes how this can be used to make the FFT algorithm calculate the frequency spectra of *two* real signals at once. One signal is placed in the real part of the time domain, while the other is placed in the imaginary part. After the FFT calculation, the spectra of the two signals are separated by an even/odd decomposition.

The Family of Fourier Transforms

Just as the DFT has a real and complex version, so do the other members of the Fourier transform family. This produces the zoo of equations shown in Table 31-1. Rather than studying these equations individually, try to understand them as a well organized and symmetrical *group*. The following comments describe the organization of the Fourier transform family. It is detailed, repetitive, and boring. Nevertheless, this is the background needed to understand theoretical DSP. Study it well.

1. Four Fourier Transforms

A time domain signal can be either *continuous* or *discrete*, and it can be either *periodic* or *aperiodic*. This defines four types of Fourier transforms: the **Discrete Fourier Transform** (discrete, periodic), the **Discrete Time Fourier Transform** (discrete, aperiodic), the **Fourier Series** (continuous, periodic), and the **Fourier Transform** (continuous, aperiodic). Don't try to understand the reasoning behind these names, there isn't any.

If a signal is discrete in one domain, it will be periodic in the other. Likewise, if a signal is continuous in one domain, will be aperiodic in the other. Continuous signals are represented by parenthesis, (), while discrete signals are represented by brackets, []. There is no notation to indicate if a signal is periodic or aperiodic.

2. Real versus Complex

Each of these four transforms has a complex version and a real version. The complex versions have a complex time domain signal and a complex frequency domain signal. The real versions have a real time domain signal and two real frequency domain signals. Both positive and negative frequencies are used in the complex cases, while only positive frequencies are used for the real transforms. The complex transforms are usually written in an exponential

form; however, Euler's relation can be used to change them into a cosine and sine form if needed.

3. Analysis and Synthesis

Each transform has an analysis equation (also called the forward transform) and a synthesis equation (also called the inverse transform). The analysis equations describe how to calculate each value in the frequency domain based on all of the values in the time domain. The synthesis equations describe how to calculate each value in the time domain based on all of the values in the frequency domain.

4. Time Domain Notation

Continuous time domain signals are called $x(t)$, while discrete time domain signals are called $x[n]$. For the complex transforms, these signals are complex. For the real transforms, these signals are real. All of the time domain signals extend from minus infinity to positive infinity. However, if the time domain is periodic, we are only concerned with a single cycle, because the rest is redundant. The variables, T and N , denote the periods of continuous and discrete signals in the time domain, respectively.

5. Frequency Domain Notation

Continuous frequency domain signals are called $X(\omega)$ if they are complex, and $ReX(\omega)$ & $ImX(\omega)$ if they are real. Discrete frequency domain signals are called $X[k]$ if they are complex, and $ReX[k]$ & $ImX[k]$ if they are real. The complex transforms have negative frequencies that extend from minus infinity to zero, and positive frequencies that extend from zero to positive infinity. The real transforms only use positive frequencies. If the frequency domain is periodic, we are only concerned with a single cycle, because the rest is redundant. For continuous frequency domains, the independent variable, ω , makes one complete period from $-\pi$ to π . In the discrete case, we use the period where k runs from 0 to $N-1$.

6. The Analysis Equations

The analysis equations operate by *correlation*, i.e., multiplying the time domain signal by a sinusoid and integrating (continuous time domain) or summing (discrete time domain) over the appropriate time domain section. If the time domain signal is aperiodic, the appropriate section is from minus infinity to positive infinity. If the time domain signal is periodic, the appropriate section is over any one complete period. The equations shown here are written with the integration (or summation) over the period: 0 to T (or 0 to $N-1$). However, any other complete period would give identical results, i.e., $-T$ to 0, $-T/2$ to $T/2$, etc.

7. The Synthesis Equations

The synthesis equations describe how an individual value in the time domain is calculated from all the points in the frequency domain. This is done by multiplying the frequency domain by a sinusoid, and integrating (continuous frequency domain) or summing (discrete frequency domain) over the appropriate frequency domain section. If the frequency domain is complex and aperiodic, the appropriate section is negative infinity to positive infinity. If the

frequency domain is complex and periodic, the appropriate section is over one complete cycle, i.e., $-\pi$ to π (continuous frequency domain), or 0 to $N-1$ (discrete frequency domain). If the frequency domain is real and aperiodic, the appropriate section is zero to positive infinity, that is, only the positive frequencies. Lastly, if the frequency domain is real and periodic, the appropriate section is over the one-half cycle containing the positive frequencies, either 0 to π (continuous frequency domain) or 0 to $N/2$ (discrete frequency domain).

8. Scaling

To make the analysis and synthesis equations undo each other, a scaling factor must be placed on one or the other equation. In Table 31-1, we have placed the scaling factors with the analysis equations. In the complex case, these scaling factors are: $1/N$, $1/T$, or $1/2\pi$. Since the real transforms do not use negative frequencies, the scaling factors are twice as large: $2/N$, $2/T$, or $1/\pi$. The real transforms also include a negative sign in the calculation of the imaginary part of the frequency spectrum (an option used to make the real transforms more consistent with the complex transforms). Lastly, the synthesis equations for the real DFT and the real Fourier Series have special scaling instructions involving $Re X(0)$ and $Re X[N/2]$.

9. Variations

These equations may look different in other publications. Here are a few variations to watch out for:

- Using f instead of ω by the relation: $\omega = 2\pi f$
- Integrating over other periods, such as: $-T$ to 0 , $-T/2$ to $T/2$, or 0 to T
- Moving all or part of the scaling factor to the synthesis equation
- Replacing the period with the fundamental frequency, $f_0 = 1/T$
- Using other variable names, for example, ω can become Ω in the DTFT, and $Re X[k]$ & $Im X[k]$ can become a_k & b_k in the Fourier Series

Why the Complex Fourier Transform is Used

It is painfully obvious from this chapter that the complex DFT is much more complicated than the real DFT. Are the benefits of the complex DFT really worth the effort to learn the intricate mathematics? The answer to this question depends on who you are, and what you plan on using DSP for. A basic premise of this book is that most practical DSP techniques can be understood and used without resorting to complex transforms. If you are learning DSP to assist in your non-DSP research or engineering, the complex DFT is probably overkill.

Nevertheless, complex mathematics is the primary language of those that specialize in DSP. If you do not understand this language, you cannot communicate with professionals in the field. This includes the ability to understand the DSP literature: books, papers, technical articles, etc. Why are complex techniques so popular with the professional DSP crowd?

Discrete Fourier Transform (DFT)

complex transform	real transform
<p><i>synthesis</i></p> $x[n] = \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$	<p><i>synthesis</i></p> $x[n] = \sum_{k=0}^{N/2} \text{Re}X[k] \cos(2\pi kn/N) - \text{Im}X[k] \sin(2\pi kn/N)$
<p><i>analysis</i></p> $X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$	<p><i>analysis</i></p> $\text{Re}X[k] = \frac{2}{N} \sum_{n=0}^{N-1} x[n] \cos(2\pi kn/N)$ $\text{Im}X[k] = \frac{-2}{N} \sum_{n=0}^{N-1} x[n] \sin(2\pi kn/N)$
<p>Time domain: x[n] is complex, discrete and periodic n runs over one period, from 0 to N-1</p> <p>Frequency domain: X[k] is complex, discrete and periodic k runs over one period, from 0 to N-1 k = 0 to N/2 are positive frequencies k = N/2 to N-1 are negative frequencies</p>	<p>Time domain: x[n] is real, discrete and periodic n runs over one period, from 0 to N-1</p> <p>Frequency domain: Re X[k] is real, discrete and periodic Im X[k] is real, discrete and periodic k runs over one-half period, from 0 to N/2</p> <p>Note: Before using the synthesis equation, the values for Re X[0] and Re X[N/2] must be divided by two.</p>

Discrete Time Fourier Transform (DTFT)

complex transform	real transform
<p><i>synthesis</i></p> $x[n] = \int_0^{2\pi} X(\omega) e^{j\omega n} d\omega$	<p><i>synthesis</i></p> $x[n] = \int_0^{\pi} \text{Re}X(\omega) \cos(\omega n) - \text{Im}X(\omega) \sin(\omega n) d\omega$
<p><i>analysis</i></p> $X(\omega) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$	<p><i>analysis</i></p> $\text{Re}X(\omega) = \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} x[n] \cos(\omega n)$ $\text{Im}X(\omega) = \frac{-1}{\pi} \sum_{n=-\infty}^{+\infty} x[n] \sin(\omega n)$
<p>Time domain: x[n] is complex, discrete and aperiodic n runs from negative to positive infinity</p> <p>Frequency domain: X(ω) is complex, continuous, and periodic ω runs over a single period, from 0 to 2π ω = 0 to π are positive frequencies ω = π to 2π are negative frequencies</p>	<p>Time domain: x[n] is real, discrete and aperiodic n runs from negative to positive infinity</p> <p>Frequency domain: Re X(ω) is real, continuous and periodic Im X(ω) is real, continuous and periodic ω runs over one-half period, from 0 to π</p>

TABLE 31-1 The Fourier Transforms

Fourier Series

complex transform	real transform
<p><i>synthesis</i></p> $x(t) = \sum_{k=-\infty}^{+\infty} X[k] e^{j 2\pi kt/T}$	<p><i>synthesis</i></p> $x(t) = \sum_{k=0}^{+\infty} \text{Re} X[k] \cos(2\pi kt/T) - \text{Im} X[k] \sin(2\pi kt/T)$
<p><i>analysis</i></p> $X[k] = \frac{1}{T} \int_0^T x(t) e^{-j 2\pi kt/T} dt$	<p><i>analysis</i></p> $\text{Re} X[k] = \frac{2}{T} \int_0^T x(t) \cos(2\pi kt/T) dt$ $\text{Im} X[k] = \frac{-2}{T} \int_0^T x(t) \sin(2\pi kt/T) dt$
<p>Time domain: x(t) is complex, continuous and periodic t runs over one period, from 0 to T</p> <p>Frequency domain: X[k] is complex, discrete, and aperiodic k runs from negative to positive infinity k > 0 are positive frequencies k < 0 are negative frequencies</p>	<p>Time domain: x(t) is real, continuous, and periodic t runs over one period, from 0 to T</p> <p>Frequency domain: Re X[k] is real, discrete and aperiodic Im X[k] is real, discrete and aperiodic k runs from zero to positive infinity</p> <p>Note: Before using the synthesis equation, the value for Re X[0] must be divided by two.</p>

Fourier Transform

complex transform	real transform
<p><i>synthesis</i></p> $x(t) = \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$	<p><i>synthesis</i></p> $x(t) = \int_0^{+\infty} \text{Re} X(\omega) \cos(\omega t) - \text{Im} X(\omega) \sin(\omega t) dt$
<p><i>analysis</i></p> $X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$	<p><i>analysis</i></p> $\text{Re} X(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} x(t) \cos(\omega t) dt$ $\text{Im} X(\omega) = \frac{-1}{\pi} \int_{-\infty}^{+\infty} x(t) \sin(\omega t) dt$
<p>Time domain: x(t) is complex, continuous and aperiodic t runs from negative to positive infinity</p> <p>Frequency domain: X(ω) is complex, continuous, and aperiodic ω runs from negative to positive infinity ω > 0 are positive frequencies ω < 0 are negative frequencies</p>	<p>Time domain: x(t) is real, continuous, and aperiodic t runs from negative to positive infinity</p> <p>Frequency domain: Re X[ω] is real, continuous and aperiodic Im X[ω] is real, continuous and aperiodic ω runs from zero to positive infinity</p>

TABLE 31-1 The Fourier Transforms

There are several reasons we have already mentioned: compact equations, symmetry between the analysis and synthesis equations, symmetry between the time and frequency domains, inclusion of negative frequencies, a stepping stone to the Laplace and z-transforms, etc.

There is also a more philosophical reason we have not discussed, something called *truth*. We started this chapter by listing several ways that the real Fourier transform is awkward. When the complex Fourier transform was introduced, the problems vanished. Wonderful, we said, the complex Fourier transform has solved the difficulties.

While this is true, it does not give the complex Fourier transform its proper due. Look at this situation this way. In spite of its abstract nature, the complex Fourier transform *properly* describes how physical systems behave. When we restrict the mathematics to be real numbers, problems arise. In other words, these problems are not *solved* by the complex Fourier transform, they are *introduced* by the real Fourier transform. In the world of mathematics, the complex Fourier transform is a greater truth than the real Fourier transform. This holds great appeal to mathematicians and academicians, a group that strives to expand human knowledge, rather than simply solving a particular problem at hand.