Design a PLL Filter When Only the Zero Resistor and Capacitor Are Adjustable

By Ken Gentile

Introduction
As described in the references, a standard procedure can be used to determine the values of \( R_o \), \( C_o \), and \( C_p \) for a second-order loop filter in a phase-locked loop (PLL). It uses open-loop bandwidth \( \omega_o \) and phase margin \( \phi_m \) as design parameters, and can be extended to third-order loop filters to determine \( R_c \) and \( C_c \) (Figure 1). The procedure solves for \( C_c \) directly and subsequently derives the remaining values.

In some cases, \( C_p \), \( R_v \), and \( C_v \) may be fixed-value components integrated within the PLL, leaving only \( R_c \) and \( C_c \) available for controlling the loop response. This nullifies the aforementioned procedure because \( C_c \) cannot be adjusted. This article proposes an alternative procedure that can be used when the value of \( C_c \) is fixed, and addresses limitations imposed by the inability to control the value of \( C_c \).

![Figure 1. Typical second-order and third-order passive loop filters.](image)

Assumptions
This loop filter design method relies on two assumptions that are typically used in third-order passive filter designs that extend a second-order loop filter design to third-order by compensating for the presence of \( R_c \) and \( C_c \) through adjustment of \( R_v \) and \( C_v \).

1. The pole frequency resulting from \( R_v \) and \( C_v \) should be at least an order of magnitude greater than \( \omega_o \) (the desired open-loop unity-gain bandwidth); specifically \( f_o \leq 0.1/(2\pi R_v C_v) \), where \( f_o = \omega_o/(2\pi) \).
2. The load of the series combination of \( R_v \) and \( C_v \) on the \( R_c \cdot C_c \cdot R_v \) network should be negligible.

Transfer Function of a Second-Order Loop Filter
A second-order loop filter has two time constants (\( T_1 \) and \( T_2 \)) associated with its components:

\[
T_2 = R_o C_o \\
T_1 = \left( \frac{C_p}{C_p + C_o} \right) T_2
\]

The loop filter’s transfer function, in terms of \( T_1 \), \( T_2 \), and \( C_v \), is important because it plays a significant role in the overall response of the PLL:

\[
H_{LF}(s) = \left( \frac{1}{C_p} \right) \left( \frac{T_1}{T_2} \right) \left( \frac{s T_2 + 1}{s(s T_1 + 1)} \right)
\]

PLL System Function
The small signal model shown in Figure 2 provides the means for formulating the PLL response and a template for analyzing phase variation at the output resulting from a phase disturbance at the input. Note that the voltage-controlled oscillator (VCO), being a frequency source, behaves like an ideal phase integrator, so its gain \( K_v \) has a 1/s factor (the Laplace transform equivalent of integration). Hence, the small signal model of a PLL has frequency dependence \( s = \sigma + j\omega \).

![Figure 2. Small signal PLL model.](image)

The closed-loop transfer function \( H_{CL}(s) \) for a PLL is defined as \( \theta_{out}/\theta_{in} \). The open-loop transfer function \( H_{OL}(s) \), defined as \( \theta_{out}/\theta_{in} \), is related to the closed-loop transfer function. It is instructive to express \( H_{OL} \) in terms of \( H_{OL} \) because the open-loop transfer function contains clues about closed-loop stability:

\[
H_{OL}(s) = -K \left( \frac{H_{LF}(s)}{sN} \right)
\]

\[
H_{CL}(s) = -N \left( \frac{H_{OL}(s)}{1-H_{OL}(s)} \right)
\]

K represents the combined gains of the phase-frequency detector (PFD), charge pump, and VCO—that is, \( K = \frac{K_p K_v}{K_{out}} \), where \( K_p \) is the charge pump current in amperes and \( K_v \) is the VCO gain in Hz/V. \( H_{OL} \), \( H_{CL} \), and \( H_{out} \) are all functions of \( s \). The negative sign in Equation 4 shows the phase inversion implied by the negative feedback to the summation node in Figure 2. Defining \( H_{OL} \) as in Equation 4 leads to subtraction in the denominator of Figure 5, which provides an intuitive explanation of closed-loop stability.

Inspection of Equation 5 reveals a potential loop stability problem. Given that \( H_{OL} \) is a function of complex frequency \( s = \sigma + j\omega \), it necessarily has frequency dependent magnitude and phase components. Therefore, if \( H_{OL} \) simultaneously exhibits unity gain and zero phase shift (or any integer multiple of \( 2\pi \) radians) for any particular value of \( s \), the denominator of \( H_{CL} \) becomes zero, the closed-loop gain becomes undefined, and the system becomes completely unstable. This implies that stability is governed by the frequency-dependent magnitude and phase characteristics of \( H_{OL} \). In fact, at the frequency for which the magnitude of \( H_{OL} \) is unity, the phase of \( H_{OL} \) must stay far enough from zero (or any integer multiple of \( 2\pi \)) to avoid a zero denominator in Equation 5.
The frequency, \( \omega_0 \), at which the magnitude of \( H_{OL} \) is unity, holds great importance. The phase of \( H_{OL} \) at \( \omega_0 \) defines the phase margin of the system \( \phi_M \). Both \( \omega_0 \) and \( \phi_M \) can be derived from \( H_{OL} \).

**Defining \( R_i \) and \( C_i \) in Terms of \( \omega_0 \) and \( \phi_M \)**

Using the design parameters \( \omega_0 \) and \( \phi_M \) to determine the values of \( R_i \) and \( C_i \) requires expressions containing those four variables and other constant terms. Start with Equation 4, because it defines \( H_{OL} \). This includes \( H_{OL} \), which includes \( R_i \) and \( C_i \) via \( T_1 \) and \( T_2 \). Since \( H_{OL} \) has magnitude and phase, it stands to reason that \( \omega_0 \) and \( \phi_M \) can be incorporated as well.

Substituting Equation 3 into Equation 4 and rearranging terms yields Equation 6, which presents \( H_{OL} \) in terms of \( T_1 \) and \( T_2 \) along with constants \( K \), \( N \), and \( C_p \):

\[
H_{OL}(s) = -\left( \frac{K}{s^2 N C_p} \right) \left( \frac{T_1}{T_2} \right) \left( \frac{sT_2 + 1}{sT_1 + 1} \right) \tag{6}
\]

Evaluation at \( s = j \omega \) yields the frequency response of \( H_{OL} \):

\[
H_{OL}(j \omega) = -\left( \frac{K}{(j \omega)^2 N C_p} \right) \left( \frac{T_1}{T_2} \right) \left( \frac{j \omega T_2 + 1}{j \omega T_1 + 1} \right) \tag{7}
\]

The \( (j \omega)^2 \) term in the denominator simplifies to \(-\omega^2\):

\[
H_{OL}(j \omega) = \left( \frac{K}{\omega^2 N C_p} \right) \left( \frac{T_1}{T_2} \right) \left( \frac{j \omega T_2 + 1}{j \omega T_1 + 1} \right) \tag{8}
\]

The magnitude and phase of \( H_{OL} \) are:

\[
|H_{OL}(j \omega)| = \left( \frac{K}{\omega^2 N C_p} \right) \left( \frac{T_1}{T_2} \right) \left( \frac{1}{1 + (\omega T_1)^2} \right) \sqrt{(1 + \omega^2 T_1 T_2)^2 + \omega^2 (T_2 - T_1)^2} \tag{9}
\]

\[
\angle H_{OL}(j \omega) = \arctan(\omega T_2) - \arctan(\omega T_1) \tag{10}
\]

Keep in mind that \( T_1 \) and \( T_2 \) are shorthand expressions for algebraic combinations of \( R_i \), \( C_i \), and \( C_p \). Evaluating Equation 9 at \( \omega = \omega_0 \) and setting \( |H_{OL}| = 1 \) defines the unity gain frequency, \( \omega_0 \), as the frequency at which the magnitude of \( H_{OL} \) is unity.

\[
1 = \left( \frac{K}{\omega_0^2 N C_p} \right) \left( \frac{T_1}{T_2} \right) \left( \frac{1}{1 + (\omega_0 T_1)^2} \right) \sqrt{(1 + \omega_0^2 T_1 T_2)^2 + \omega_0^2 (T_2 - T_1)^2} \tag{11}
\]

Similarly, evaluating Equation 10 at \( \omega = \omega_0 \) and setting \( \angle H_{OL} = \phi_M \) defines the phase margin, \( \phi_M \), as the phase of \( H_{OL} \) at frequency \( \omega_0 \) (the unity gain frequency).

\[
\phi_M = \arctan(\omega_0 T_2) - \arctan(\omega_0 T_1) \tag{12}
\]

It is a trivial matter to expand Equation 11 and Equation 12 by substituting Equation 1 for \( T_1 \) and Equation 2 for \( T_2 \), which brings \( R_i \) and \( C_i \) into the equations. Hence, we have succeeded in relating \( \omega_0 \) and \( \phi_M \) to the variables \( R_i \) and \( C_i \) along with constants \( K \), \( N \), and \( C_p \).

Simultaneously solving the resulting equations for \( R_i \) and \( C_i \) is no trivial task. The symbolic processor available in MathCad can solve the two simultaneous equations, but arccos must be substituted for arctan. This transformation enables the symbolic processor to solve for \( R_i \) and \( C_p \) yielding the following solution sets (\( R_{OA} \), \( C_{OA} \); \( R_{OB} \), \( C_{OB} \); \( R_{OC} \), \( C_{OC} \); and \( R_{OD} \), \( C_{OD} \)). See the Appendix for details on transforming Equation 12 to use the arccos function.

\[
R_{OA} = \frac{\omega_0 K N \sqrt{1 - \cos^2(\phi_M)}}{K^2 + 2 K C_p N \omega_0^2 \cos(\phi_M) + (C_p N \omega_0^2)^2}
\]

\[
R_{OB} = -\left( \frac{\omega_0 K N \sqrt{1 - \cos^2(\phi_M)}}{K^2 + 2 K C_p N \omega_0^2 \cos(\phi_M) + (C_p N \omega_0^2)^2} \right)
\]

\[
R_{OC} = \frac{\omega_0 K N \sqrt{1 - \cos^2(\phi_M)}}{K^2 - 2 K C_p N \omega_0^2 \cos(\phi_M) + (C_p N \omega_0^2)^2}
\]

\[
R_{OD} = -\left( \frac{\omega_0 K N \sqrt{1 - \cos^2(\phi_M)}}{K^2 - 2 K C_p N \omega_0^2 \cos(\phi_M) + (C_p N \omega_0^2)^2} \right)
\]

\[
C_{OA} = -\left( \frac{K^2 + 2 K C_p N \omega_0^2 \cos(\phi_M) + (C_p N \omega_0^2)^2}{N \omega_0^2 (C_p N \omega_0^2 + K \cos(\phi_M))} \right)
\]

\[
C_{OB} = -\left( \frac{K^2 + 2 K C_p N \omega_0^2 \cos(\phi_M) + (C_p N \omega_0^2)^2}{N \omega_0^2 (C_p N \omega_0^2 + K \cos(\phi_M))} \right)
\]

\[
C_{OC} = -\left( \frac{K^2 - 2 K C_p N \omega_0^2 \cos(\phi_M) + (C_p N \omega_0^2)^2}{N \omega_0^2 (C_p N \omega_0^2 - K \cos(\phi_M))} \right)
\]

\[
C_{OD} = -\left( \frac{K^2 - 2 K C_p N \omega_0^2 \cos(\phi_M) + (C_p N \omega_0^2)^2}{N \omega_0^2 (C_p N \omega_0^2 - K \cos(\phi_M))} \right)
\]
This result is problematic because the goal was to solve for $R_0$ and $C_0$ given $\omega_0$ and $\phi_M$, but this indicates four possible $R_0$, $C_0$ pairs instead of a unique $R_0$, $C_0$ pair. However, closer inspection of the four results leads to a single solution set as follows.

Note that in the context of modeling a PLL, all of the variables in the above equations possess positive values, including $\cos(\phi_M)$ because $\phi_M$ is constrained to values between 0 and $\pi/2$. As a result, $C_0$ and $R_0$ are clearly negative quantities.

Therefore, solution sets $R_0A$, $C_0A$ and $R_0B$, $C_0B$ are immediately ruled out because negative component values are not acceptable. The $R_0C$, $C_0C$ and $R_0D$, $C_0D$ results require further analysis, however.

Note that the four equations involving $R_0C$, $C_0C$ and $R_0D$, $C_0D$ possess the common factor:

$$K^2 = 2KC_PN\omega_0^2 \cos(\Phi_M) + (C_PN\omega_0^2)^2 \text{ (13)}$$

Closer inspection reveals that Expression 13 has the form $a^2 - (2ac)\cos(\beta) + c^2$. Equating this with the arbitrary quantity $b^2$, yields:

$$b^2 = a^2 + c^2 - (2ac)\cos(\beta) \text{ (14)}$$

Equation 14, the Law of Cosines, relates $a$, $b$, and $c$ as the lengths of the three sides of a triangle with $\beta$ being the interior angle of the vertex opposite side $b$. Since $b^2$ is the square of the length of one side of a triangle, it must be a positive quantity, which implies the right side of Equation 14 must also be positive. Thus, Expression 13 must be a positive quantity, which means the denominator of $R_0$ is positive. The numerator of $R_0$ is also positive, therefore $R_0$, must be negative, which rules out the $R_0D$, $C_0D$ solution set. This leaves only the $R_0C$, $C_0C$ pair as a contender for the simultaneous solution of Equation 11 and Equation 12.

$$R_0 = \frac{\omega_0KN\sqrt{1 - \cos^2(\Phi_M)}}{K^2 - 2KC_PN\omega_0^2 \cos(\Phi_M) + (C_PN\omega_0^2)^2} \text{ (15)}$$

$$C_0 = \frac{K^2 - 2KC_PN\omega_0^2 \cos(\Phi_M) + (C_PN\omega_0^2)^2}{N\omega_0^2(K \cos(\Phi_M) - C_PN\omega_0^2)} \text{ (16)}$$

**Constraints on $R_0$ and $C_0$**

Although Equation 15 and Equation 16 are contenders for the simultaneous solution of Equation 11 and Equation 12, they are only valid if they result in positive values for both $R_0$ and $C_0$. Close inspection of $R_0$ shows it to be positive—its numerator is positive, because the range of $\cos(x)$ is 0 to 1—and its denominator is the same as Expression 13, which was previously shown to be positive. The numerator of $C_0$ is also the same as Expression 13, so $C_0$ is positive as long as its denominator satisfies the following condition:

$$K \cos(\Phi_M) > C_PN\omega_0^2 \text{ (17)}$$

This is depicted graphically in Figure 3, in which the left and right sides of Equation 17 are each equated to $y$ (blue and green curves) with the horizontal axis sharing $\omega_0$ and $\phi_M$. The intersection of the two curves marks the boundary condition for $\omega_0$ and $\phi_M$. The condition under which Equation 17 is true appears as the red arc. The portion of the horizontal axis beneath the red arc defines the range of $\phi_M$ that ensures $C_0$ is positive. Note the point on the horizontal axis directly below the intersection of the blue and green curves establishes $\phi_M,\text{MAX}$, the maximum value of $\phi_M$ to ensure $C_0$ is positive.

$$\Phi_{M,\text{MAX}} = \arccos \left( \frac{C_PN\omega_0^2}{K} \right) \text{ radians} \text{ (18)}$$

Equation 18 requires that $C_PN\omega_0^2$ be less than $K$ in order to satisfy the constraints of the arccos function for $\phi_M,\text{MAX}$ between 0 and $\pi/2$. This establishes $\omega_0,\text{MAX}$, the upper limit on $\omega_0$ to ensure $C_0$ is positive.

$$\omega_{0,\text{MAX}} = \sqrt{\frac{K}{C_PN}} \text{ radians/s} \text{ (19)}$$

**Figure 3. Constraint on $C_0$ denominator.**

**Compensating for $R_2$ and $C_2$ (Third-Order Loop Filter)**

In the case of a third-order loop filter, components $R_2$ and $C_2$ introduce additional phase shift, $\Delta \phi$, relative to the second-order loop filter:

$$\Delta \Phi = -\arctan(\omega_0 R_2 C_2) \text{ (20)}$$

To deal with this excess phase shift, subtract it from the desired value of $\phi_M$.
The incorporation of this form of HLF into the HOL and Simulation Results and phase margin results deviate only slightly from the design acceptable results. In fact, the simulated open-loop bandwidth PLL with a third-order loop filter. The simulations all have the parameters (R0 and C0). Such simulations reveal the frequency response and phase margin associated with maximum of ¼ of the value dictated by Equation 19 yields

\[ \omega \]

The procedure for determining R0 and C0 assumed a second-order loop filter, but is extendible to third-order loop filter designs by adjusting the desired phase margin (\( \phi_0 \)) to a new value (\( \phi_{\text{M,new}} \)) per Equation 21, yielding a new upper bound (\( \phi_{\text{M,new}} \)) per Equation 22.

Although simulations using a second-order loop filter validated Equation 15 and Equation 16, validating the equations that extend the design procedure to third-order loop filter designs requires a redefinition of the loop filter response, \( H(s) \), to include \( R_2 \) and \( C_2 \) as follows:

\[ H_L(s) = \frac{sR_0C_0 + 1} {s^2R_0R_2C_0C_2C_P + sR_2C_0C_2 + sR_0C_0C_P + sR_2C_2C_P + sR_0C_2C_0 + C_0 + C_2 + C_P} \]

Conclusion
This article demonstrates using open-loop unity-gain bandwidth (\( \omega_0 \)) and phase margin (\( \phi_0 \)) as design parameters for second-order or third-order loop filters when only components \( R_0 \) and \( C_0 \) are adjustable. Simulation of a PLL with a second-order loop filter using \( R_0 \) and \( C_0 \) yields an exact match to the theoretical frequency response of \( H_{\text{OL}} \) and the resulting phase margin, thereby validating the equations. The parameters \( \omega_0 \) and \( \phi_0 \) have upper bounds for a second-order loop filter per Equation 19 and Equation 18, respectively.

The procedure for determining \( R_0 \) and \( C_0 \) assumed a second-order loop filter, but is extendible to third-order loop filter designs for both Simulation 1 and Simulation 2 use \( \omega_0 = 100 \text{ Hz} \), which is near the calculated upper limit of 124.8 Hz (\( \omega_{\text{M,MAX}} \)). As such, Simulation 1 and Simulation 2 deviate from the design parameter values (\( \omega_0 \) and \( \phi_0 \)) by nearly 10%. On the other hand, Simulation 3 and Simulation 4 use \( \omega_0 = 35 \text{ Hz} \), which is approximately ¼ the upper limit. As expected, Simulation 3 and Simulation 4 hold much closer to the design parameters (\( \omega_0 \) and \( \phi_0 \)), yielding an error of only about 1%.

Table 1 summarizes the simulation results and also includes the calculated values of \( R_0 \), \( C_0 \), \( \omega_0 \), \( \phi_0 \), \( \omega_{\text{M,MAX}} \), and \( \phi_{\text{M,MAX}} \) for the given design parameters, \( \omega_0 \) and \( \phi_0 \). Note that for the purpose of comparison it would be preferable for both Simulation 1 and Simulation 3 to use \( \phi_0 = 80^\circ \), but Simulation 1 must satisfy the constraint imposed by Equation 22 of \( \phi_0 < 48^\circ \) (hence the choice of 42\(^\circ \)).
Table 1: Simulation Summary

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<th>Simulation 1</th>
<th>Simulation 2</th>
<th>Simulation 3</th>
<th>Simulation 4</th>
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<td>100 Hz</td>
<td>30°</td>
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<td>124.8 Hz</td>
<td>124.8 Hz</td>
<td>124.8 Hz</td>
</tr>
<tr>
<td>$\phi_{\text{M,MAX}}$</td>
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<td>48.0°</td>
<td>84.8°</td>
<td>84.8°</td>
</tr>
</tbody>
</table>

Figure 4 and Figure 5 show the open- and closed-loop response for each simulation.

Appendix—Converting the Discontinuous Arctan Function to the Continuous Arccos Function

Equation 10 demonstrates that the angle $\phi$ is the difference between angle $\theta_2$ and angle $\theta_1$, where $\theta_2 = \arctan(\omega T_2)$ and $\theta_1 = \arctan(\omega T_1)$. Furthermore, $\omega T_2$ is expressible as $x/1$ and $\omega T_1$ as $y/1$:

$$\Phi = \theta_2 - \theta_1 = \arctan \left( \frac{x}{1} \right) - \arctan \left( \frac{y}{1} \right)$$

This implies the geometric relationship shown in Figure 6, with $\theta_1$ and $\theta_2$ defined by the triangles of Figure 6 (b) and (a), respectively. Figure 6 (c) combines these two triangles to show $\phi$ as the difference between $\theta_1$ and $\theta_2$.

The Law of Cosines relates an interior angle ($\theta$) of a triangle to the lengths of the three sides of the triangle ($a$, $b$, and $c$) as follows:

$$c^2 = a^2 + b^2 + 2ab \cos(\theta)$$

($\theta$ is the angle opposite side $c$)

Applying the Law of Cosines to angle $\phi$ in Figure 6 (c) yields:

$$(x - y)^2 = \left( \sqrt{1+x^2} \right)^2 + \left( \sqrt{1+y^2} \right)^2 - 2\sqrt{1+x^2}\sqrt{1+y^2} \cos \Phi$$
Figure 6. Geometric representation of Equation 10.

Solving for $\Phi$:

$$\Phi = \arccos\left(\frac{1 + xy}{\sqrt{(1 + x^2)(1 + y^2)}}\right)$$

But, $x/1 = \omega T_2$ and $y/1 = \omega T_1$, allowing $\phi$ to be expressed in terms of $T_1$ and $T_2$.

$$\Phi = \arccos\left(\frac{1 + \omega^2 T_1 T_2}{\sqrt{1 + (\omega T_2)^2}[1 + (\omega T_1)^2]}\right)$$

References


MT-086: Fundamentals of Phase Locked Loops (PLLs). PLLs/PLLs with Integrated VCOs.

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Also by this Author:

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