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Quantization Noise: An Expanded Derivation of the Equation, *SNR* = 6.02 N + 1.76 dB

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IN THIS MINI TUTORIAL

The steps are shown for how the equation, signal-to-noise-ratio (SNR) = 6.02 N + 1.76 dB is derived. The mathematical derivation steps are highlighted.

INTRODUCTION

This tutorial describes three distinct stages for the derivation process.

1. The ideal analog-to-digital converter (ADC) transfer function equation and manipulation.
2. The root mean square (rms) derivation from integration.
3. The SNR equation derivation for obtaining the $SNR = 6.02\ N + 1.76$ dB value.

This mathematical tutorial expands and enhances the derivation version presented in MT-001.

IDEAL ADC TRANSFER FUNCTION EQUATION AND MANIPULATION

The ideal ADC transfer function is shown in Figure 1(A). The digital (binary) output values are represented by the y-axis, and the analog inputs are represented by the x-axis. The diagonal staircase represents the quantized value of the analog input signal. The dashed line through the staircase represents their mid-points.

Figure 1(B) represents the quantization noise of an ideal N-bit ADC for a ramp input signal. The quantization error of 1 LSB peak-to-peak can be approximated by an uncorrelated saw tooth waveform having a maximum peak-to-peak swing of q, from $q/2$ to $-q/2$. Note that t_1 and t_2 are points in time and are used at a later stage in the derivation. This signal is the difference between the quantized output signal (solid) and the analog input signal (dashed) shown in Figure 1(A).

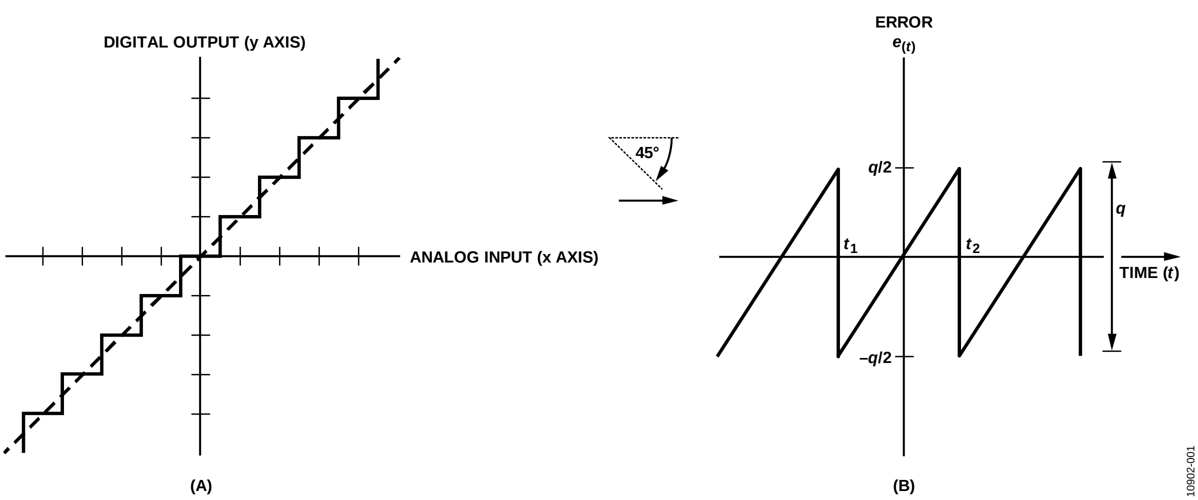

Figure 1. Ideal ADC Transfer Function (A) and Ideal N-Bit ADC Quantized Noise (B)

The equation of a line is given by

$y = mx + c$

where:
y represents the value on the y-axis.
m is the slope.
x is the value on the x-axis.
c is the intersection point where the line passes through the y-axis at $x = 0$.

Therefore, when the equation of a line is applied in Figure 2, when c is at $x = 0$, $y = 0$ (that is, at the origin). The error equation for $e(t)$ is

$$e_{(t)} = st + 0 \text{ or simply } e_{(t)} = st \quad (1)$$

where:
$e_{(t)}$ is the quantized error.
s is the slope.
t is the time.

This is simply the equation of a straight line

$y = mx + c$

where:
$y = e(t)$.
$m = s$.
$x = t$.
$c = 0$.

The error, e(t), swings between −q/2 and +q/2 for $t_1 < t < t_2$.

At Time t_1 and t_2 in Figure 1, the error $e(t)$ is given by

$$e(t_1) = \frac{-q}{2} = st_1$$

$$t_1 = \frac{-q}{2s} \quad (2)$$

$$e(t_2) = \frac{q}{2} = st_2$$

$$t_2 = \frac{q}{2s} \quad (3)$$

Equation 2 and Equation 3 can now relabel the graph for e(t) as shown in Figure 2.

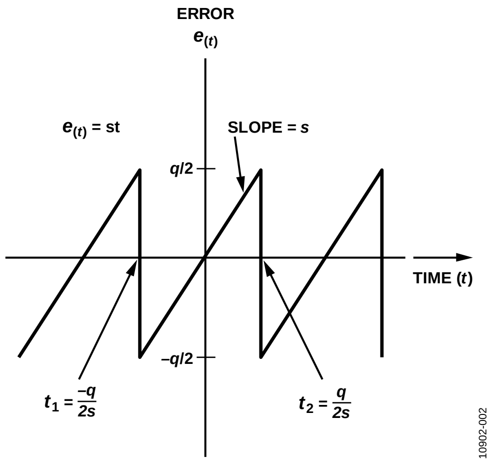

Figure 2. Substitution of Values for t_1 and t_2

RMS DERIVATION

The root mean square (rms) derivation can now be evaluated by integration and substitution. Figure 3(A) shows the period, T, over which the integration is performed. The mean square of e(t) is shown in Figure 3(B).

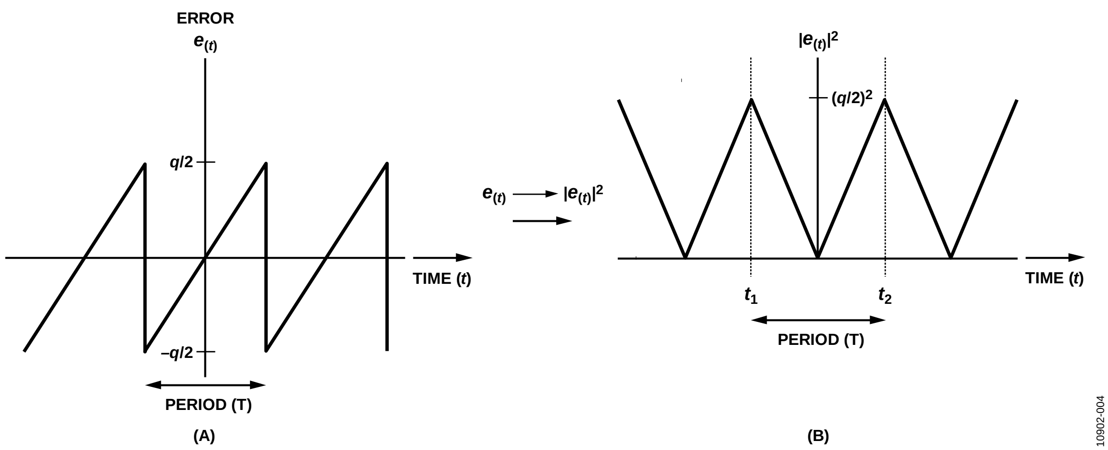

Figure 3. Defining the Period T (A) and Squaring the Error e(t) Function (B)

The mean square error e(t), is computed, over the period T, where the Time t is defined by Equation 2 and Equation 3,

$$t_1 = \frac{-q}{2s}, \qquad t_2 = \frac{+q}{2s},$$

Equating and defining the base for the period T in Figure 3 as,

$$T = t_2 - t_1$$

$$T = \frac{q}{2s} + \frac{q}{2s}$$

$$\therefore\ T = \frac{q}{s} \tag{4}$$

mean square error,

$$\bar{e}^2{}_{(t)} = \int_{t_1}^{t_2} \frac{(st)^2}{T} dt = \frac{q^2}{12} \tag{5}$$

is derived as follows by evaluating the mean square error from integration:

$$\bar{e}^2{}_{(t)} = \int_{t_1}^{t_2} \frac{\frac{(st)^2}{1}}{\frac{q}{s}} dt = \int_{t_1}^{t_2} \frac{(st)^2}{1} \times \frac{s}{q} dt$$

$$\bar{e}^2{}_{(t)} = \frac{s}{q}\int_{t_1}^{t_2} (st)^2\, dt \tag{6}$$

$$= \frac{s}{q}\int_{t_1}^{t_2} s^2 t^2\, dt = \frac{s^3}{q}\int_{t_1}^{t_2} t^2\, dt$$

$$= \frac{s^3}{q}\left[\frac{t^3}{3}\right]_{t_1}^{t_2}$$

$$= \frac{s^3}{q}\left[\left.\frac{t^3}{3}\right|^{t_2} - \left.\frac{t^3}{3}\right|_{t_1}\right]$$

Substituting for upper and lower limits for t_2 and t_1,

$$= \frac{s^3}{q}\left[\frac{\left(\frac{q}{2s}\right)^3}{3} + \frac{\left(\frac{q}{2s}\right)^3}{3}\right] = \frac{s^3}{q}\left[2\frac{\left(\frac{q}{2s}\right)^3}{3}\right]$$

$$= \frac{s^3}{q} \times 2\left[\frac{\frac{q^3}{8s^3}}{\frac{3}{1}}\right] = \frac{s^3}{q} \times 2\left[\frac{q^3}{8s^3} \times \frac{1}{3}\right]$$

$$= \frac{s^3}{q} \times 2 \times \frac{q^3}{8s^3} \times \frac{1}{3}$$

$$= \frac{\cancel{s^3}}{\cancel{q}} \times \cancel{2} \times \frac{\cancel{q^3}}{\cancel{8}_4\, \cancel{s^3}} \times \frac{1}{3}$$

$$= \frac{q^2}{4} \times \frac{1}{3}$$

Therefore, the derived mean square error,

$$\bar{e}^2{}_{(t)} = \frac{q^2}{12} \qquad \text{QED}$$

Evaluating the root mean square error e(t), can be found from

$$\sqrt{\left(\bar{e}^2{}_{(t)}\right)} = \sqrt{\frac{q^2}{12}} = \frac{q}{\sqrt{12}} = \frac{q}{\sqrt{(4 \times 3)}} = \frac{q}{2\sqrt{3}}$$

Therefore, the rms quantized error e(t),

$$\sqrt{\left(\bar{e}^2{}_{(t)}\right)} = \frac{q}{2\sqrt{3}} \tag{7}$$

The theoretical signal-to-noise ratio can be calculated, assuming an average full-scale (FS) sinewave, $\overline{V_{(t)}}$, as the input signal where

$$\overline{V_{(t)}} = \frac{q2^N}{2}\sin(2\pi f t) \tag{8}$$

For a sinewave to be converted to an rms value, simply multiply by $\frac{1}{\sqrt{2}}$ or by 0.707. Hence, $\overline{V_{(t)}}$ x $\frac{1}{\sqrt{2}}$. Therefore, the root mean square of the input sinewave is given as

$$\sqrt{\bar{V}^2{}_{(t)}} = \frac{q2^N}{2\sqrt{2}}\sin(2\pi f t) \tag{9}$$

SNR DERIVATION

The SNR equations in dB, where the 6.02 N + 1.76 dB, can now be derived from this point.

From Equation 9, the maximum amplitude occurs when sine (90°) = 1. The rms (FS) sinewave input V(t) signal can now be written as

$$\sqrt{\bar{V}^2{}_{(t)}} = \frac{q2^N}{2\sqrt{2}} \tag{10}$$

The rms signal-to-noise ratio, for an ideal N-bit converter (Equation 10, for example) with respect to the rms value of quantization noise (for example, Equation 7), hence (Equation 10)/(Equation 7), can now be computed in dB as

$$SNR = 20\, log_{10}\, \frac{RMS\ value\ of\ FS\ input}{RMS\ value\ of\ quantization\ noise} \tag{11}$$

$$SNR = 20\, log_{10}\left[\frac{\sqrt{\bar{V}^2{}_{(t)}}}{\sqrt{\left(\bar{e}^2{}_{(t)}\right)}}\right]$$

$$= 20\, log_{10}\left[\frac{\frac{q2^N}{2\sqrt{2}}}{\frac{q}{2\sqrt{3}}}\right]$$

$$= 20\, log_{10}\left[\frac{q2^N}{2\sqrt{2}} \times \frac{2\sqrt{3}}{q}\right]$$

$$= 20\, log_{10}\left[\frac{\cancel{q}2^N}{\cancel{2}\sqrt{2}} \times \frac{\cancel{2}\sqrt{3}}{\cancel{q}}\right]$$

$$= 20\, log_{10}\left[\frac{2^N}{1} \times \frac{\sqrt{3}}{\sqrt{2}}\right]$$

$$= 20\, log_{10}\left[2^N \times \sqrt{\frac{3}{2}}\right]$$

$$= 20\, log_{10}\, [2]^N + 20\, log_{10}\left[\frac{3}{2}\right]^{\frac{1}{2}}$$

$$= N \times 20\, log_{10}\,(2) + \frac{1}{2} \times 20\, log_{10}\left(\frac{3}{2}\right)$$

$$= N \times 20 \times 0.301 + 10 \times 0.176$$

$$\therefore SNR = \underline{6.02\,N + 1.76\,dB} \qquad \text{QED}$$

where N is the bit resolution of an ADC.

The derivation shows that the 6.02 factor in the equation is derived from $20log_{10}(2)$ and the 1.76 dB term is derived from $10log_{10}\left(\frac{3}{2}\right)$.

SUMMARY

This equation is an approximation, which assumes that the quantization error is not correlated to the input signal. This assumption is true in most cases where N > 6 and the input signal is not an exact submultiple of the sampling frequency. This case is discussed in more detail in MT-001.

The noise term calculated to determine SNR in the equation is the noise measured over the Nyquist bandwidth, dc to one-half the sampling frequency. If the bandwidth of interest is less than one-half the sampling frequency, then a correction factor must be applied as discussed in MT-001.

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REVISION HISTORY

8/12—Revision 0: Initial Version